Matter collineations of Spacetime Homogeneous Gödel-type Metrics

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Abstract. The spacetime homogeneous Gödel-type spacetimes which have four classes of metrics are studied according to their matter collineations. The obtained results are compared with Killing vectors and Ricci collineations. It is found that these spacetimes have infinite number of matter collineations in degenerate case, i.e. $\det(T_{ab}) = 0$, and do not admit proper matter collineations in non-degenerate case, i.e. $\det(T_{ab}) \neq 0$. The degenerate case has the new constraints on the parameters m and w which characterize the causality features of the Gödel-type spacetimes.

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1. Introduction

Let M be a spacetime manifold with Lorentz metric g of signature (+ - - -). The manifold M and the metric g are assumed smooth (C^∞). Throughout this article, the usual component notation in local charts will often be used, and a covariant derivative with respect to the symmetric connection Γ associated with the metric g will be denoted by a semicolon and a partial derivative by a comma.

Einstein's field equations (EFEs) in local coordinates are given by

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = \kappa T_{ab}, \tag{1}$$

where G_{ab} are the components of the Einstein tensor, R_{ab} those of the Ricci and T_{ab} of the matter (energy-momentum) tensor. Also, $R = g^{ab} R_{ab}$ is the Ricci scalar, and it is assumed that $\kappa = -1$ and $\Lambda = 0$ for simplicity. In general relativity (GR) theory, the Einstein tensor G_{ab} plays a significant role, since it relates the geometry of spacetime

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to its source. The GR theory, however, does not prescribe the various forms of matter, and takes over the energy-momentum tensor T_{ab} from other branches of physics.

The EFEs (1), whose fundamental constituent is the spacetime metric g_{ab}, are highly nonlinear partial differential equations, and therefore it is very difficult to obtain their exact solutions. Symmetries of the geometrical/physical relevant quantities of the GR theory are known as *collineations*. In general, these can be represented as $\pounds_\xi \mathcal{A} = \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are the geometric/physical objects, ξ is the vector field generating the symmetry, and $\pounds_\xi$ signifies the Lie derivative operator along the vector field ξ.

Matter symmetries provide a different, and more physically based, approach to symmetries of spacetimes. The physical interest of these symmetries can be discussed as follows. For a given distribution of matter, the contribution of gravitational potential satisfying EFEs is the principal aim of all investigations in gravitational physics. This has been achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. In an attempt to study the geometric and physical properties of the electromagnetic fields, different types of collineations have been investigated [1],[2] along with many other interesting results. It is seen that for a null electromagnetic field a motion does not imply Maxwell collineation and conversely. Symmetries of the energy-momentum tensor (also called matter collineations defined in the following) provide conservation laws on matter fields. These enable us to know how the physical fields, occupying in certain region of spacetimes, reflect the symmetries of the metric [3]. In other words, given the metric tensor of a spacetime, one can find symmetry for the physical fields describing the material content of that spacetime.

A one-parameter group of conformal motions generated by a *conformal Killing vector* (CKV) ξ is defined as [4]

$$\pounds_\xi g_{ab} = 2\psi g_{ab}, \tag{2}$$

where $\psi = \psi(x^a)$ is a conformal factor. If $\psi_{;ab} \neq 0$, the CKV is said to be *proper*. Otherwise, ξ reduces to the special *conformal Killing vector* (SCKV) if $\psi_{;ab} = 0$, but $\psi_{,a} \neq 0$. Other subcases are *homothetic vector* (HV) if $\psi_{,a} = 0$ and *Killing vector* (KV) if $\psi = 0$.

Using Eq. (2), we find from Eq. (1) that

$$\pounds_\xi T_{ab} = -2\psi_{;ab} + 2g_{ab}\Box\psi, \tag{3}$$

where $\Box$ is the Laplacian operator defined by $\Box\psi \equiv g^{cd}\psi_{;cd}$. Therefore, for a KV, or HV, or SCKV we have

$$\pounds_\xi T_{ab} = 0 \quad \Leftrightarrow \quad \pounds_\xi G_{ab} = 0, \tag{4}$$

or in component form

$$T_{ab,c}\xi^c + T_{ac}\xi^c_{,b} + T_{cb}\xi^c_{,a} = 0. \tag{5}$$

A vector field ξ satisfying Eq. (4) or (5) on M is called a *matter collineation* (MC). Therefore, we define a *proper* MC to be an MC which is not an KV, *or* an HV, *or* an SCKV. Since the Ricci tensor arises naturally from the Riemann curvature tensor (with

components R^a_{bcd} and where $R_{ab} \equiv R^c_{acb}$) and hence from the connection, the study of *Ricci collineation* (RC) defined by $\pounds_\xi R_{ab} = 0$ has a natural geometrical significance [5]-[12]. Mathematical similarities between the Ricci and energy-momentum tensors mean many techniques for their study should show some similarities. From the physical viewpoint explained in the above, a study of MCs, i.e. to look into the set of solutions to Eq.(5) seems more relevant. In this direction, some papers have recently been appeared on MCs [13]-[19]. In addition, since energy-momentum tensor T_{ab} is more fundamental in the study of dynamics of fluid spacetimes of GR, the remainder of this paper will be concerned with MCs.

Recently, Carot *et al.* [13] and Hall *et al.* [14] have noticed some important results about the Lie algebra of MCs. These are the following:

a. The set of all MCs on M is a vector space, but it may be infinite dimensional and may not be a Lie algebra. If T_{ab} is non-degenerate, i.e. $det(T_{ab}) \neq 0$, the Lie algebra of MCs is finite dimensional. If T_{ab} is degenerate, i.e. $det(T_{ab}) = 0$, we cannot guarantee the finite dimensionality of the MCs.

b. If the energy-momentum tensor T_{ab} is everywhere of rank 4 then it may be regarded as a metric on M. Then, it follows by a standard result that the family of MCs is, in fact, a Lie algebra of smooth vector fields on M of finite dimension ≤ 10 (and $\neq 9$).

c. If a vector field ξ on M is a symmetry of *all* the gravitational field sources, then one could require the Eq. (4) (for the *non-vacuum* sources) and $\pounds_\xi C^a_{bcd} = 0$ (for the *vacuum* sources), where C^a_{bcd} are Weyl curvature tensor components.

Throughout this paper, it is used the form (5) of MCs without imposing any restriction on the energy-momentum tensor.

The plan of the paper is as follows. In the next section we shall describe the Gödel-type spacetimes with some general results. In Section 3, we shall write down MC Eqs. (5) for the metric given by (6), and solve them for spacetime homogeneous Gödel-type metrics. Finally, we shall provide a brief summary and discussion of the results obtained.

2. Gödel-type metrics

In 1949, Gödel found a solution of Einstein's field equations with cosmological constant for incoherent matter with rotation [20]. It is certainly the best known example of a cosmological model which makes it apparent that GR does not exclude the existence of closed timelike world lines, despite its Lorentzian character which leads to the local validity of the causality principle. Gödel's cosmological solution has a well-recognized importance which has, to a large extent, motivated the investigations on rotating cosmological Gödel-type spacetimes and on causal anomalies within the framework of GR [21, 22]. In natural cylindrical coordinates $x^a = (t, r, \phi, z)$, $a = 0, 1, 2, 3$, the Gödel-type metrics are given by

$$ds^2 = [dt + H(r)d\phi]^2 - dr^2 - D^2(r)d\phi^2 - dz^2. \tag{6}$$

In 1980, Raychaudhuri and Thakurta [23] has given the necessary conditions that a metric of Gödel-type is spacetime homogeneous (ST homogeneous, hereafter). Three years later, Rebouças and Tiomno [24] proved that these conditions

$$\frac{D''}{D} = const \equiv m^2, \tag{7}$$

$$\frac{H'}{D} = const \equiv -2\omega \tag{8}$$

are both necessary and sufficient. However, in both articles [23, 24], the study of ST homogenity is limited in that only time-independent KV fields were considered [25]. Finally, these conditions were proved to be the necessary and sufficient conditions for a Gödel-type manifold to be ST homogeneous without assuming any such simplifying hypothesis [26]. The above results for Gödel-type manifolds can be collected together as follows :

Theorem 1: The necessary and sufficient conditions for a four-dimensional Riemannian Gödel-type manifold to be locally homogeneous are those given by Eqs. (7) and (8).

Theorem 2: The four-dimensional homogeneous Riemannian Gödel-type manifolds admit group of isometry G_r with

(i) $r = 5$ if $m^2(>0) \neq 4\omega^2$ with $\omega \neq 0$, or when $m^2 = 0$ and $\omega \neq 0$, or when $m^2 \equiv -\mu^2(<0)$ and $\omega \neq 0$;

(ii) $r = 6$ if $m^2 \neq 0$ and $\omega = 0$;

(iii) $r = 7$ if $m^2(>0) = 4\omega^2$ and $\omega \neq 0$.

Theorem 3: The four-dimensional homogeneous Riemannian Gödel-type manifolds are locally characterized by two independent parameters m^2 and ω: the pair of (m^2, ω) identically specify locally equivalent manifolds.

Now, we shall be concerned with irreducible set of isometrically nonequivalent homogeneous Gödel-type metrics which was given in Refs. 28 and 29. These distinguish the following four classes of metrics according to:

Class I : $m^2 > 0, \omega \neq 0$. For this case, the general solution of Eqs. (7) and (8) can be written as

$$H(r) = \frac{2\omega}{m^2}\left[1 - \cosh(mr)\right] \quad and \quad D(r) = \frac{1}{m}\sinh(mr). \tag{9}$$

Class II : $m^2 = 0, \omega \neq 0$. For this case, the general solution of Eqs. (7) and (8) is

$$H(r) = -\omega r^2 \quad and \quad D(r) = r, \tag{10}$$

where only the essential parameter ω appears.

Class III : $m^2 \equiv -\mu^2 < 0, \omega \neq 0$. Similarly, the integration of the conditions for homogeneity Eqs. (7) and (8) leads to

$$H(r) = \frac{2\omega}{\mu^2}\left[\cos(\mu r) - 1\right] \quad and \quad D(r) = \frac{1}{\mu}\sin(\mu r). \tag{11}$$

Class IV : $m^2 \neq 0, \omega = 0$. We refer to the manifolds of this class as degenerated Gödel-type manifolds, since the cross term in the line element, related to the rotation

ω in the Gödel model, vanishes. By a trivial coordinate transformation one can make $H = 0$ with $D(r)$ given, respectively, by Eqs. (9) or (11) depending on whether $m^2 > 0$ or $m^2 \equiv -\mu^2 < 0$.

If $m^2 = \omega = 0$, then the line element (6) is clearly Minkowskian. Therefore, this particular case has not been included in this study. Also, it is noted that the condition $m^2 = 2w^2$ defines nothing but the original Gödel model, which is known to violate the causality principle. Rebouças *et al.* [26] and Calvão *et al.* [27] have found that the causality features of the Gödel-type spacetimes depend upon the the two independent parameters given above, i.e. m and w. They have shown that for $0 \leq m^2 < 4w^2$, there exists only *one* noncausal region; for $m^2 \geq 4w^2$ there are no closed timelike curves, in which a completely causal and ST homogeneous Gödel model corresponds to the limiting case $m^2 = 4w^2$; for $m^2 < 0$ there are infinite number of alternating causal and noncausal regions.

The KV fields as well as corresponding Lie algebra of each class are given in Appendix B [24],[25],[29]. Firstly, Hall and Costa [30] have pointed out that the original Gödel metric does not admit HVs and laterly, this result is extended to the ST homogeneous Gödel-type spacetimes [8]. Recently, the proper CKVs and complete conformal algebra of a Gödel-type spacetime have been computed in [31] applying their method. The RCs and contracted RCs of ST homogeneous Gödel-type spacetimes are studied by Melfo *et al.* [8]. In this study, we provide a complete solution of the MC equations for ST homogeneous Gödel-type spacetimes.

3. Matter Collineation Equations and Their Solutions

For the ST homogeneous Gödel-type metrics, the non-vanishing components of T_{ab} become

$$T_{00} = 3w^2 - m^2, \tag{12}$$
$$T_{02} = (3w^2 - m^2)H, \tag{13}$$
$$T_{11} = w^2, \tag{14}$$
$$T_{22} = (3w^2 - m^2)H^2 + w^2 D^2, \tag{15}$$
$$T_{33} = m^2 - w^2, \tag{16}$$

(see Appendix A). Thus, $det(T_{ab}) = w^4(m^2 - w^2)(3w^2 - m^2)D^2$. If $w^2 \neq 0$, $m^2 \neq m^2$, *and* $m^2 \neq 3w^2$, then T_{ab} is non-degenerate. When $w^2 = 0$ *or* $m^2 = w^2$ *or* $m^2 = 3w^2$, then T_{ab} is degenerate.

For the ST homogeneous Gödel-type metric (6), writing equation (5) in expanded form, we obtain the following MC equations :

$$w^2 \xi^1_{,r} = 0, \tag{17}$$
$$(m^2 - w^2)\xi^3_{,z} = 0, \tag{18}$$
$$w^2 \xi^1_{,z} + (m^2 - w^2)\xi^3_{,r} = 0, \tag{19}$$
$$(3w^2 - m^2)F_{,t} = 0, \tag{20}$$

$$(3w^2 - m^2)F_{,z} + (m^2 - w^2)\xi^3_{,t} = 0, \tag{21}$$
$$(3w^2 - m^2)H\, F_{,z} + w^2 D^2 \xi^2_{,z} + (m^2 - w^2)\xi^3_{,\phi} = 0, \tag{22}$$
$$(3w^2 - m^2)H\left[F_{,\phi} + H'\xi^1\right] + w^2 D^2\left[\xi^2_{,\phi} + \frac{D'}{D}\xi^1\right] = 0, \tag{23}$$
$$(3w^2 - m^2)\left[F_{,\phi} - 2wD\xi^1\right] + w^2 D^2 \xi^2_{,t} = 0, \tag{24}$$
$$(3w^2 - m^2)\left[F_{,r} + 2wD\xi^2\right] + w^2 \xi^1_{,t} = 0, \tag{25}$$
$$(3w^2 - m^2)H\,\left[F_{,r} + 2wD\xi^2\right] + w^2 D^2 \xi^2_{,r} + w^2 \xi^1_{,\phi} = 0. \tag{26}$$

where F is defined as

$$F = \xi^0 + H\xi^2. \tag{27}$$

Now, in the following, we consider degenerate and non-degenerate cases of MCs, respectively.

3.1. Degenerate Case

In this case, we have the possibilities in which the energy-momentum tensor T_{ab} becomes degenerate, (**i**) $w = 0, m^2 \neq 0$; (**ii**) $m^2 = w^2, w \neq 0$; (**iii**) $m^2 = 3w^2, w \neq 0$. It is seen from these possibilities that we do *not* have degenerate energy-momentum tensor T_{ab} in the Class II.

Case (i). This case corresponds to the Class IV case, and since one can make $H = 0$ by a trivial coordinate transformation, we obtain

$$T_{00} = -m^2 = -T_{33}, \qquad T_{02} = T_{11} = T_{22} = 0. \tag{28}$$

The solution of the MC equations gives

$$\xi^0 = c_1 + c_3 z,\ \xi^1 = \xi^1(t,r,\phi,z),\ \xi^2 = \xi^2(t,r,\phi,z),\ \xi^3 = c_2 + c_3 t, \tag{29}$$

which can be written as follows

$$\xi_{(1)} = \partial_t, \quad \xi_{(2)} = \partial_z, \quad \xi_{(3)} = z\partial_t + t\partial_z, \tag{30}$$
$$\xi_{(4+i)} = \xi^1(t,r,\phi,z)\partial_r + \xi^2(t,r,\phi,z)\partial_\phi, \tag{31}$$

where "i" is an arbitrary positive number. In order to construct a closed abelian algebra for vectors given in (30) and (31), we find that $\xi^1 = \xi^1(r,\phi)$ and $\xi^2 = \xi^2(r,\phi)$. Thus, we have infinite number of MCs.

Case (ii). For this case, the Classes I and III are to be taken into consideration. The condition $m^2 = w^2$ give rise to

$$T_{00} = 2w^2,\ T_{02} = 2w^2 H,\ T_{11} = w^2,\ T_{22} = w^2(2H^2 + D^2),\ T_{33} = 0. \tag{32}$$

Then, the MC equations yield the solutions

$$\xi^0 = c_1 - c_2\left(\frac{2}{w}D' + H\right),\ \xi^1 = 0,\ \xi^2 = c_2,\ \xi^3 = \xi^3(t,r,\phi,z), \tag{33}$$

which can be stated as

$$\xi_{(1)} = \partial_t, \quad \xi_{(2)} = \partial_\phi - \left(\frac{2}{w}D' + H\right)\partial_t, \quad \xi_{(3+i)} = \xi^3(t,r,\phi,z)\partial_z. \tag{34}$$

Therefore, we have also infinite number of MCs.

Case (iii). In this case, the MC equations reveal only for Classes I and III that

$$\xi^0 = \xi^0(t,r,\phi,z), \quad \xi^1 = c_1\cos\phi + c_2\sin\phi, \tag{35}$$

$$\xi^2 = \frac{D'}{D}(-c_1\sin\phi + c_2\cos\phi) + c_3, \quad \xi^3 = c_4. \tag{36}$$

Then, it follows from these results that

$$\xi_{(1)} = \xi^0(t,r,\phi,z)\partial_t, \tag{37}$$

$$\xi_{(2)} = \partial_z + \xi^0(t,r,\phi,z)\partial_t, \tag{38}$$

$$\xi_{(3)} = \partial_\phi + \xi^0(t,r,\phi,z)\partial_t, \tag{39}$$

$$\xi_{(4)} = \xi^0(t,r,\phi,z)\partial_t + \cos\phi\partial_r - \frac{D'}{D}\sin\phi\partial_\phi, \tag{40}$$

$$\xi_{(5)} = \xi^0(t,r,\phi,z)\partial_t - \sin\phi\partial_r - \frac{D'}{D}\cos\phi\partial_\phi, \tag{41}$$

where we have used the following property

$$D^2\left(\frac{D'}{D}\right)' = -1, \tag{42}$$

which is valid for ST homogeneous Gödel-type metrics only. Thus, in this case one also finds infinite number of MCs. For this case, the constraint $m^2 = 3w^2$ yields

$$T_{11} = w^2, \quad T_{22} = w^2D^2, \quad T_{33} = 2w^2 \qquad T_{00} = T_{02} = 0. \tag{43}$$

3.2. Non-degenerate Case

In this case, since the energy-momentum tensor T_{ab} is non-degenerate, we assume that $w^2 \neq 0, m^2 \neq w^2$ and $m^2 \neq 3w^2$. From Eqs. (17)-(23), it follows that

$$\begin{aligned}
\xi^0 \quad &= -H\,\xi^2 + \frac{z}{3w^2-m^2}(c_1\phi + c_2) + P(r,\phi),\\
\xi^1 \quad &= Q(t,\phi),\\
\xi^2 \quad &= \frac{z}{w^2D^2}\left[c_1(t-\phi\,H) - c_2\,H - c_3(m^2-w^2)\right] - \frac{3W^2-m^2}{w^2})\frac{H}{D^2}P(r,\phi)\\
&\quad + \left[\frac{2}{w}(3w^2-m^2)\frac{H}{D} - \frac{D'}{D}\right]\int Q(t,\phi)d\phi + R(t,r),\\
\xi^3 \quad &= \frac{t}{w^2-m^2}(c_1\phi + c_2)t + c_3\phi + c_4,
\end{aligned}$$

where $P(r,\phi), Q(t,\phi)$, and $R(t,r)$ are integration functions. Then, using Eqs. (24) and (25), we find for Classes I, II, and III that $c_1 = c_2 = c_3 = 0$ and

$$P = 2wD\int K_1(\phi)d^2\phi + 2w\phi\,D\,c_5 + L_1(r), \tag{44}$$

$$Q = \int K_1(\phi)d\phi + c_5, \quad R = L_2(r), \tag{45}$$

$$L_{1,r} - \frac{2}{w}(3w^2-m^2)\frac{H}{D}L_1 + 2wDL_2 = 0, \tag{46}$$

where $K_1(\phi), L_1(r)$, and $L_2(r)$ are integration functions, and we have assumed $m^2 \neq 4w^2$. Then, from the remaining Eq. (26), we obtain that $c_5 = 0$ and

$$K_{1,\phi\phi} - \epsilon K_1 = 0, \tag{47}$$

$$L_{2,r} - (3 - \frac{m^2}{w^2})\left(\frac{H}{D^2} L_1\right)_{,r} = 0, \tag{48}$$

$$D^2 \left(\frac{D'}{D}\right)' = \epsilon. \tag{49}$$

For the ST homogeneous Gödel-type metrics in cylindrically coordinates, ϵ takes the value -1 only, while $\epsilon = 0$ case coincides with the original Gödel metric [20]. The solution of the above equations (46)-(49) for Class I is given as follows

$$\begin{aligned}
\xi^0 &= \frac{H}{D}(k_1 \cos\phi + k_2 \sin\phi) - \frac{2w}{m} k_3 + k_4,\\
\xi^1 &= k_1 \sin\phi - k_2 \cos\phi,\\
\xi^2 &= \frac{D'}{D}(k_1 \cos\phi + k_2 \sin\phi) + m k_3,\\
\xi^3 &= k_5,
\end{aligned}$$

where k_1, k_2, k_3, k_4, and k_5 are constants. Thus, these correspond to the KVs for this class (see Appendix B). Similarly, using Eqs.(46)-(49), one can obtain the MCs for Classes II and III which are only the KVs.

When $m^2 = 4w^2$, Eqs. (25) and (26) yield that $c_1 = c_2 = c_3 = 0$, and

$$P = 2wD \int K_1(\phi) d^2\phi + L_1(r), \quad R = L_2(r), \tag{50}$$

$$Q_{,tt} + m^2 Q = m^2 \int K_1 d\phi, \tag{51}$$

$$Q_{,\phi\phi} + Q = K_{1,\phi} + \int K_1 d\phi, \tag{52}$$

$$K_{1,\phi\phi} + K_1 = 0, \tag{53}$$

$$L_{1,r} + 2w\frac{H}{D} L_1 + 2wD L_2 = 0, \tag{54}$$

$$\frac{H}{D^2} L_1 + L_2 = c_5. \tag{55}$$

Now, after solving the above equations and rearranging the appearing constants, it follows that

$$\begin{aligned}
\xi^0 &= \frac{H}{D}\left[k_1 \sin(mt + \phi) - k_2 \cos(mt + \phi) + k_3 \cos\phi + \sin\phi\right] - k_6 + k_7,\\
\xi^1 &= k_1 \cos(mt + \phi) + k_2 \sin(mt + \phi) + k_3 \sin\phi - k_4 \cos\phi,\\
\xi^2 &= -\frac{1}{D}\left[k_1 \sin(mt + \phi) - k_2 \cos(mt + \phi)\right]\\
&\quad + \frac{D'}{D}(k_3 \cos\phi + k_4 \sin\phi) + m k_6,\\
\xi^3 &= k_5
\end{aligned}$$

which are the KVs given in Class I for case $m^2 = 4w^2$.

4. Conclusions

In this paper, we have obtained MCs for ST homogeneous Gödel-type metrics. As a conclusion, the RC and MC equations for these metrics deserve some remarks. First, for these spacetimes, we have only degenerate case of the RC equations, because of $\det(R_{ab}) = 0$. Also, ξ^3 does not appear in the RC equations, and therefore any metric of the form (6) admits a proper RC of the form [8] $\xi = f(t, r, \phi, z)\partial_z$ which represents that we have infinite number of RCs. Second, for the ST homogeneous Gödel-type metrics, we have both degenerate and non-degenerate cases of the MC equations given in Sec.3. In the MC classification of these spacetimes according to the nature of the energy-momentum tensor, we find that when the energy-momentum tensor is degenerate, subsection 3.1, then there are infinite number of MCs. In this subsection, we have found the new constraints on the parameters m and w, which are $m^2 = w^2$, case (ii), and $m^2 = 3w^2$, case (iii). These new constraints are in the interval $0 < m^2 < 4w^2$, which means that there exist only one noncausal region. Also, the cases (ii) and (iii) correspond to the Classes I and III only. For the case (ii), we have from Eq. (A7) that the Ricci scalar vanishes. Therefore, the MCs and RCs are obviously coincide for this case. Furthermore, in subsection 3.2, we proceed to deal with the MCs of the ST homogeneous Gödel-type spacetimes in which the energy-momentum tensor is non-degenerate. In this subsection, we find that all MCs of ST homogeneous Gödel-type metrics are the KVs given in Appendix B, that is, these spacetimes given by the Classes I-IV do not admit proper MCs.

Although it is noted in Ref. 13 that when rank of T_{ab} is *one*, the physical interest in degenerate case is only limited to dust fluids (perfect fluids with $p = 0$), *or* null fluids (radiation and null Einstein-Maxwell fields), it seems that when rank of T_{ab} is *three*, i.e. the cases (ii) and (iii), it may be possible to find the physically significant spacetimes with degenerate energy-momentum tensor.

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Appendix A

For the Gödel-type space-times, we find the following results for the non-vanishing components of the fully-covariant curvature tensor R_{abcd}

$$R_{0101} = \left(\frac{H'}{2D}\right)^2, \qquad R_{0202} = \frac{H'^2}{4}, \tag{A1}$$

$$R_{0112} = -\frac{H}{2}\left[\frac{D}{H}\left(\frac{H'}{D}\right)' + \frac{1}{2}\left(\frac{H'}{D}\right)^2\right], \tag{A2}$$

$$R_{1212} = H^2\left[\frac{D}{H}\left(\frac{H'}{D}\right)' + \left(\frac{H'}{2D}\right)^2 + 3\left(\frac{H'}{2H}\right)^2\right] - DD'', \tag{A3}$$

the Ricci tensor R_{ab}

$$R_{00} = \frac{1}{2}\left(\frac{H'}{D}\right)^2, \qquad R_{11} = \frac{D''}{D} - \frac{1}{2}\left(\frac{H'}{D}\right)^2, \tag{A4}$$

$$R_{02} = -\frac{H}{2}\left[\frac{D}{H}\left(\frac{H'}{D}\right)' + \left(\frac{H'}{D}\right)^2\right], \tag{A5}$$

$$R_{22} = -H^2\left[\frac{D}{H}\left(\frac{H'}{D}\right)' + \frac{1}{2}\left(\frac{H'}{D}\right)^2 + \frac{1}{2}\left(\frac{H'}{H}\right)^2\right] + DD'', \tag{A6}$$

and the scalar curvature R

$$R = -\frac{2D''}{D} + \frac{1}{2}\left(\frac{H'}{D}\right)^2 \tag{A7}$$

respectively. By using the above expressions, the non-vanishing components of the Einstein tensor G_{ab} are the following

$$G_{00} = \frac{D''}{D} - 3\left(\frac{H'}{2D}\right)^2, \qquad G_{11} = -\left(\frac{H'}{2D}\right)^2, \tag{A8}$$

$$G_{02} = -\frac{H}{2}\left[\frac{D}{H}\left(\frac{H'}{D}\right)' - 2\frac{D''}{D} + \frac{3}{2}\left(\frac{H'}{D}\right)^2\right], \tag{A9}$$

$$G_{22} = -H^2\left[\frac{D}{H}\left(\frac{H'}{D}\right)' - \frac{D''}{D} + 3\left(\frac{H'}{2D}\right)^2 + \left(\frac{H'}{2H}\right)^2\right], \tag{A10}$$

$$G_{33} = -\frac{D''}{D} + \left(\frac{H'}{2D}\right)^2. \tag{A11}$$

Appendix B

The KVs of the ST homogeneous Gödel-type spacetimes presented in the following were first obtained in Refs. 24 and 25, can also be derived as a particular case of KVs calculated in Ref. 29. Thus, Linearly independent KVs for the Gödel-type space-times in Class I-Class IV are given as

Class I : $m^2 > 0, \omega \neq 0$. When $m^2 \neq 4\omega^2$,

$$\begin{aligned}
&\mathbf{K}_1 = \partial_t, \quad \mathbf{K}_2 = \partial_z, \quad \mathbf{K}_3 = \frac{2\omega}{m}\partial_t - m\partial_\phi, \\
&\mathbf{K}_4 = -\frac{H}{D}\sin\phi\partial_t + \cos\phi\partial_r - \frac{D'}{D}\sin\phi\partial_\phi, \\
&\mathbf{K}_5 = -\frac{H}{D}\cos\phi\partial_t - \sin\phi\partial_r - \frac{D'}{D}\cos\phi\partial_\phi
\end{aligned} \tag{B1}$$

The Lie algebra has the following nonvanishing commutators:

$$[\mathbf{K}_3, \mathbf{K}_4] = -m\mathbf{K}_5, \quad [\mathbf{K}_4, \mathbf{K}_5] = m\mathbf{K}_3, \quad [\mathbf{K}_5, \mathbf{K}_3] = -m\mathbf{K}_4.$$

It should be noticed that the expressions for all KVs are time-independent. When $m^2 = 4\omega^2$,

$$\begin{aligned}
&\mathbf{K}_1 = \partial_t, \quad \mathbf{K}_2 = \partial_z, \quad \mathbf{K}_3 = \partial_t - m\partial_\phi, \\
&\mathbf{K}_4 = -\frac{H}{D}\sin\phi\partial_t + \cos\phi\partial_r - \frac{D'}{D}\sin\phi\partial_\phi, \\
&\mathbf{K}_5 = -\frac{H}{D}\cos\phi\partial_t - \sin\phi\partial_r - \frac{D'}{D}\cos\phi\partial_\phi, \\
&\mathbf{K}_6 = -\frac{H}{D}\cos(mt+\phi)\partial_t + \sin(mt+\phi)\partial_r + \frac{1}{D}\cos(mt+\phi)\partial_\phi, \\
&\mathbf{K}_7 = -\frac{H}{D}\sin(mt+\phi)\partial_t - \cos(mt+\phi)\partial_r + \frac{1}{D}\sin(mt+\phi)\partial_\phi,
\end{aligned} \tag{B2}$$

where $m = +2\omega$. The corresponding Lie algebra has the following nonvanishing commutators:

$$\begin{aligned}
&[\mathbf{K}_3, \mathbf{K}_4] = -m\mathbf{K}_5, \quad [\mathbf{K}_4, \mathbf{K}_5] = m\mathbf{K}_3, \quad [\mathbf{K}_5, \mathbf{K}_3] = -m\mathbf{K}_4, \\
&[\mathbf{K}_1, \mathbf{K}_6] = -m\mathbf{K}_7, \quad [\mathbf{K}_6, \mathbf{K}_7] = m\mathbf{K}_1, \quad [\mathbf{K}_7, \mathbf{K}_1] = -m\mathbf{K}_6.
\end{aligned}$$

Class II : $m^2 = 0, \omega \neq 0$. In this Class, the KVs are

$$\begin{aligned}
&\mathbf{K}_1 = \partial_t, \quad \mathbf{K}_2 = \partial_z, \quad \mathbf{K}_3 = \partial_\phi, \\
&\mathbf{K}_4 = -\omega\, r\,\sin\phi\partial_t - \cos\phi\partial_r + \frac{1}{r}\sin\phi\partial_\phi, \\
&\mathbf{K}_5 = -\omega\, r\,\cos\phi\partial_t + \sin\phi\partial_r + \frac{1}{r}\cos\phi\partial_\phi
\end{aligned} \tag{B3}$$

For these KVs, the Lie algebra is

$$[\mathbf{K}_3, \mathbf{K}_4] = \mathbf{K}_5, \quad [\mathbf{K}_3, \mathbf{K}_5] = -\mathbf{K}_4, \quad [\mathbf{K}_4, \mathbf{K}_5] = 2\omega\mathbf{K}_1.$$

Class III : $m^2 \equiv -\mu^2 < 0, \omega \neq 0$. In this Class, the KVs are

$$\begin{aligned}
&\mathbf{K}_1 = \partial_t, \quad \mathbf{K}_2 = \partial_z, \quad \mathbf{K}_3 = \frac{2\omega}{\mu}\partial_t + \mu\,\partial_\phi, \\
&\mathbf{K}_4 = -\frac{H}{D}\sin\phi\,\partial_t + \cos\phi\partial_r - \frac{D'}{D}\sin\phi\partial_\phi, \\
&\mathbf{K}_5 = -\frac{H}{D}\cos\phi\partial_t - \sin\phi\partial_r - \frac{D'}{D}\cos\phi\partial_\phi
\end{aligned} \tag{B4}$$

where the Lie algebra is given by

$$[\mathbf{K}_3, \mathbf{K}_4] = \mu\mathbf{K}_5, \quad [\mathbf{K}_3, \mathbf{K}_5] = -\mu\mathbf{K}_4, \quad [\mathbf{K}_4, \mathbf{K}_5] = \mu\mathbf{K}_3.$$

Class IV : $m^2 \neq 0, \omega = 0$. In this class, $H = 0$ and $D(r) = \frac{1}{m} sinh(mr)$ for $m^2 > 0$, or $D(r) = \frac{1}{\mu} sin(\mu r)$ for $m^2 \equiv -\mu^2 < 0$.

$$\begin{aligned} &\mathbf{K}_1 = \partial_t, \quad \mathbf{K}_2 = \partial_z, \quad \mathbf{K}_3 = z\,\partial_t + t\partial_z, \\ &\mathbf{K}_4 = \cos\phi\partial_r - \frac{D'}{D}\sin\phi\partial_\phi, \\ &\mathbf{K}_5 = -\sin\phi\partial_r - \frac{D'}{D}\cos\phi\partial_\phi, \quad \mathbf{K}_6 = \partial_\phi \end{aligned} \tag{B5}$$

with the nonvanishing commutators

$$\begin{aligned} &[\mathbf{K}_1, \mathbf{K}_3] = \mathbf{K}_2, \quad [\mathbf{K}_2, \mathbf{K}_3] = \mathbf{K}_1, \\ &[\mathbf{K}_4, \mathbf{K}_5] = -m^2\mathbf{K}_6, \quad [\mathbf{K}_5, \mathbf{K}_6] = \mathbf{K}_4, \quad [\mathbf{K}_6, \mathbf{K}_4] = \mathbf{K}_5. \end{aligned}$$

It is noticed that when $m = \omega = 0$ the line element (6) is clearly Minkowskian. It admits the Poincaré group.

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